1. Calculate the volume

\[ V(R) = \int \ldots \int \Theta \left( R - \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \right) dx_1 dx_2 \ldots dx_n \quad (1) \]

and the surface area

\[ S(R) = dV(R)/dR \]

of \( n \)-dimensional sphere of radius \( R \)

(a) Show that

\[ V(R) = C_n R^n \]

where \( C_n \) is the constant to be determined below

(b) Calculate \( C_n \) and verify its value for \( n = 2 \) and \( 3 \)

Hint: Consider an integral:

\[ \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp \left[ -\lambda \left( x_1^2 + x_2^2, \ldots, +x_n^2 \right) \right] dx_1 dx_2 \ldots dx_n \]

and calculate it. Then express the integral in spherical coordinates as:

\[ \int_0^{\infty} \exp(-\lambda R^2) S(R) dr \]

2. Consider a classical ideal gas of \( N \) atoms confined in volume \( V \). The number \( N \) is very large and all quantities of the order of \( \ln N/N \) are negligible. The Hamiltonian of the system reads:

\[ H = \sum_{i=1}^{N} \frac{p_{ix}^2 + p_{iy}^2 + p_{iz}^2}{2m} \]

where \( p_{i\alpha} \) is the cartesian component of the momentum of \( i^{th} \) atom and \( m \) is the mass. Use the results of problem ?? to calculate:

(a) Number of states \( \Gamma(E,V) \) with energy less or equal to \( E \)
3. If the ideal gas is placed in an external harmonic-oscillator trapping potential the Hamiltonian would become:

\[
H = \sum_{i=1}^{N} \left[ \frac{p_{ix}^2 + p_{iy}^2 + p_{iz}^2}{2m} + \frac{1}{2} m \omega^2 \left( x_i^2 + y_i^2 + z_i^2 \right) \right]
\]

Here \{x_i, y_i, z_i\} are the coordinates of \(i\)th atom and \(\omega\) characterizes strength of the trapping potential. Use the results of problem ?? to calculate:

(a) The number of states with energy less or equal to \(E\)
(b) The entropy of the system

4. Prepare a Mathematica notebook describing neutrino oscillations presented in Lecture 4. Using the time-dependent state vector

\[
|\psi(t)\rangle = \cos \theta \, |\nu_1\rangle + \sin \theta \exp(-i\omega t) \, |\nu_2\rangle,
\]

where \(\omega = (E_1 - E_2)/\hbar\) and \(|\nu_i\rangle\) is a mass eigenstate with energy \(E_i\)

perform the calculations as follows.

(a) Compute (analytically) the density matrices of the pure states:

\[
\hat{\rho}(t) = |\psi(t)\rangle \langle \psi(t)|
\]

\[
\hat{\rho}_\mu = |\mu\rangle \langle \mu|
\]

\[
\hat{\rho}_\tau = |\tau\rangle \langle \tau|
\]

(b) Prove that all these matrices are idempotent, i.e. \(\hat{\rho}^2 = \hat{\rho}\)

(c) Compute the probabilities to find neutrinos in either \(\mu\) or \(\tau\) states:

\[
P_\tau(t) = \text{Tr}[\hat{\rho}_\tau \cdot \hat{\rho}(t)]
\]

\[
P_\mu(t) = \text{Tr}[\hat{\rho}_\mu \cdot \hat{\rho}(t)]
\]

(d) Compute the expectation values of Pauli operators \(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\) in isospin space:

\[
\langle \hat{\sigma}_i(t) \rangle = \text{Tr}[\hat{\sigma}_i \cdot \hat{\rho}(t)]
\]
(e) Plot $P_\mu(t), P_\tau(t)$ for $\omega = 2\pi$ and $\theta = \pi/8, \pi/4, \pi/3$ in the time interval $0 \leq t \leq 5$

(f) Plot $\langle \hat{\sigma}_i(t) \rangle$ for $\omega = 2\pi$ and $\theta = \pi/8, \pi/4, \pi/3$ in the time interval $0 \leq t \leq 5$